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Structural System Reliability Quantification Using Multipoint Function Approximations

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In structural problems, when dealing with uncertainties, the failure probability of the structure is estimated subject to a particular performance criterion. However, when the failure of a structural system is governed by multiple failure criteria, all of the measures have to be considered in the failure probability estimation. These failure criteria are usually correlated, and the accuracy of the estimated structural failure probability highly depends on the ability to model the joint failure surface. For example, in an aircraft structure, the stresses in each of the members of a wing can be posed as limit-state functions, along with the displacements and the natural frequencies of the wing. There are no criteria to disregard one limit state over the other, or to convert the system reliability problem into component reliability (dealing with displacement, stress, and frequency individually). Each failure criterion is modeled as a limit-state function for the reliability analysis, which is an implicit function of the random variables. The evaluation of this limit state often requires an expensive finite element simulation or a computational fluid dynamics simulation. Therefore, to predict the failure probability of a structural system efficiently, function approximations for the limit states are considered. An accurate way of defining highly nonlinear functions is presented using a new class of approximations. These approximations are used in conjunction with the Monte Carlo simulation to estimate the structural failure probability. Numerical examples are presented to show the applicability of the proposed method.

Introduction

A STRUCTURE typically consists of many components, each of which has the potential to fail, and the individual component failure might lead to structural failure. Even in simple structures composed of just one element, various failure modes such as bending action, buckling, axial stress, temperature, frequency, etc., may exist and be relevant to the solution. The composition of many elements in structures is referred to as a "structural system," and a system may be subject to many forms of loads, either single or in various combinations. Therefore, the reliability analysis of structural systems will involve consideration of multiple and perhaps correlated limit states that can be defined in any discipline. Each limit state is an implicit function and requires expensive computer time to evaluate the function value and the gradients required for the reliability analysis. Therefore, the presence of multiple limit states increases the computational effort involved in the failure probability estimation process.

The system failure probability is an integration of the joint probability density function (PDF) over the joint failure domain obtained by the intersection of all of the limit states. The joint PDF is an implicit function and can be evaluated numerically using Monte Carlo simulation. However, this would require a large number of exact function evaluations, which would come from expensive finite element analysis (FEA) or computational fluid dynamics (CFD) simulations. The cost involved in the simulations renders the Monte Carlo simulation unsuitable for most of the practical structural reliability problems. Therefore, alternative methods are required to estimate the structural failure probability.

The most commonly used classifications for the structural systems are 1) series systems and 2) parallel systems.[1] The series systems are those in which, even if one component fails to perform satisfactorily, the whole system will fail. This is also called a weakest-link model. Because every component is required to function satisfactorily for the system to be reliable, the failure probability of every component is estimated using various approximation techniques. In this paper, the first-order reliability method (FORM)[2] is used to estimate the component failure probability to compare the results with the proposed method. It is evident that a statically determinate structure is a series system because the failure of any one of its members implies the failure of the structure.

In the case of a parallel system, the system survives even if one component has failed. The system fails to function satisfactorily only when every component of the system has failed to function satisfactorily. Parallel systems are sometimes referred to as redundant systems. There are two types of redundancies: active redundancy and passive redundancy. Active redundancy occurs when redundant elements actively participate in structural behavior, even during low loading. Passive redundancy occurs when the redundant elements do not come into play until the structure has suffered a sufficient degree of degradation or failure of its elements. A system that is a combination of both series and parallel components is called a mixed system.

In structural system reliability analysis, the bound methods and numerical integration methods have practical significance. If the components of the system are assumed independent, then the system failure can be obtained easily. However, in practical problems, the failure conditions depend on the same random variables; therefore, the components are correlated. Cornell[3] has developed bounds on the system failure probability for systems subjected to multiple failure modes. The upper bound on the system failure was obtained by assuming perfectly correlated components, and this is obtained as follows, with the upper bound on P_f equaling

$$\sum_{i}^{n} [\text{component} P_f] \tag{1}$$

The lower bound is obtained by assuming statistically independent components, with the lower bound on P_f equaling

$$\max[\text{component} P_f] \tag{2}$$

where n is the number of failure modes. If all of the components are perfectly independent, then the failure probability bounds can be obtained by the preceding lower and upper bound formulations.

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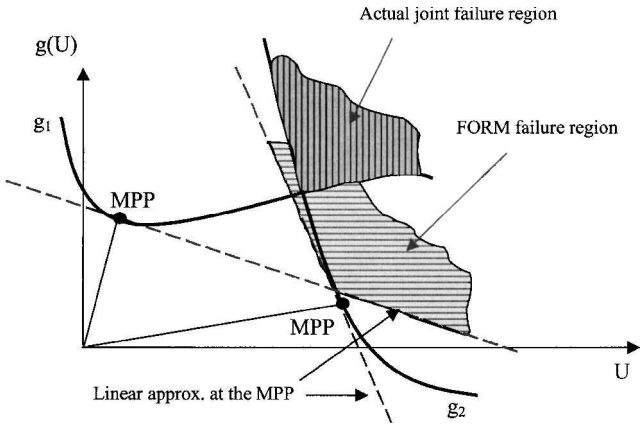

Fig. 1 Joint failure region.

However, the component P_f has to be quantified accurately to obtain an accurate system reliability bound. To achieve better accuracy compared to first- and second-order methods, the component failure probability can be determined by using the high-quality approximation techniques developed by Penmetsa et al. in earlier work.[4]

In FORM, the limit state is approximated with a linear function at the most probable failure point (MPP). The MPP is the point on the limit state that is nearest to the origin in a standard normal space that is obtained by performing Rosenblatt transformation[5] to the random variables. Because of rotational symmetry and exponential decay of the probability density in the standard normal space, the MPP has the highest likelihood of failure among all points in the failure domain. Therefore, the neighborhood of the MPP makes a major contribution to the failure probability integral. This property is the basis for FORM, which constructs an approximation to the failure probability integral by using the tangent plane at the MPP as the integration boundary. The second-order reliability method (SORM) improves on this approximation with a quadratic surface using the second-order gradients. In system failure probability, the probability of failure of each of the individual limit states is typically estimated using either the FORM or the SORM.

As shown in Fig. 1, the first-order approximation at the MPP would result in an erroneous approximation of a nonlinear limit-state function. This, in turn, would result in a poor approximation of the joint failure region. Therefore, a high-quality approximation is required to capture the information of the limit states around the MPP and the joint failure region.

The method of narrow bounds, presented by Ditlevsen[6] for the system failure probability, had wider applicability due to its high accuracy. These bounds considered the correlation between each of the two failure modes, making the results more physically reasonable. With this method, the system failure probability can be expressed from the bounds of first- or second-order joint probabilities. However, these bounds are quite accurate only when the limit states are of linear form. In situations where this assumption is not valid, alternative procedures have to be developed to estimate the failure probability.

To improve the accuracy of the Ditlevsen's[6] bounds, both theoretically and practically, Feng[7] has developed a method using third-order joint probability for computing the system failure probability. This method uses the first-, second-, and third-order joint failure probabilities to estimate the failure probability accurately. For problems where the second- and third-order joint probabilities can be estimated accurately using shorter computer run times, the resulting accuracy is high.

The system failure probability obtained using Ditlevsen's[6] method when correlations among the failure modes are less than 60% has narrow bounds; otherwise, it has wide bounds. Similarly, the bounds obtained by Feng's[7] method are accurate when the joint failure probabilities can be estimated accurately. However, in most circumstances the formulas for computing the second- and third-order joint probabilities have large errors. Therefore, Song[8] has proposed a method using numerical integration in a reduced domain of failure region. Song proposed to reduce the failure domain by a factor of safety index in every direction and later used numerical integration in the reduced domain. This method reduces the number of actual simulations and gives accurate results for a low number of failure modes. The computer time of this method increases exponentially with the number of failure modes. Therefore, when the structure has many failure modes, this method cannot be directly used for computing the system failure probability. He has proposed a method to deal with this drawback; however, that method required second- and third-order joint failure probabilities. The alternative method uses the FORM failure probability, which introduces errors.

When dealing with highly nonlinear problems with a large number of nonnormal random variables and implicit limit-state functions, both the FORM and SORM approximations fail to give accurate results. Therefore, better approximations such as two-point adaptive nonlinear approximations [(TANA2)[9] or (TANA3)[10]] have to be used to approximate the limit-state functions. The approximations capture the information of the limit state accurately in the vicinity of the MPP. When dealing with multiple limit states, information about the MPP of each limit state is vital for the accurate estimation of the system failure probability. Therefore, the two-point approximation is used as a local approximation at each of the MPPs of every limit state; then the multipoint approximations (MPA) are constructed. This MPA retains the information of each of the failure surfaces and constructs a joint failure domain. Because this joint failure domain is constructed using accurate approximations of the individual failure domains, it can be integrated using the Monte Carlo simulation technique to obtain the system failure probability. The reduction in computational cost of system reliability prediction significantly helps in the preliminary design of large-scale multifunctional structures.

Proposed Method

The Monte Carlo approach (with a significant number of simulations) is the most reliable method for component/system reliability prediction. To improve the efficiency of Monte Carlo simulations, the required analysis can be performed on an approximate limit-state function. When the function value and the gradient information at the MPP are matched for an approximate function, it will be able to capture the critical failure region around the MPP. This approximation can be used in the analysis algorithm. The wider range of applicability of MPAs compared to single-point approximations make it more suitable to replace the actual function in the Monte Carlo simulations.

The MPA can be written using the following general formulation:

$$\tilde{F}(X) = \sum_{k=1}^{K} W_k(\boldsymbol{X})\tilde{F}_k(\boldsymbol{X}) \tag{3}$$

where $\tilde{F}_k(\boldsymbol{X})$ is a two-point local approximation and W_k is a weighting function that adjusts the contribution of $\tilde{F}_k(\boldsymbol{X})$ to $\tilde{F}(\boldsymbol{X})$ in Eq. (3). The evaluation of this weighting function involves the selection of a blending function and a power index m. The procedural details for evaluating the weighting function are discussed in the Appendix.

Naturally, the accuracy of a local approximation is one of the primary factors on which the quality of the MPA is dependent. Therefore, TANA2 were used as local approximations to construct the MPA of each limit-state function. TANA2 can capture the information of the limit state accurately in the vicinity of the data points. The MPA retains the information of each of the failure surface without increasing the computational effort. Because this joint failure domain is constructed using more accurate approximations of the individual failure domains, it can be integrated using the Monte Carlo simulation technique to obtain the system failure probability.

Most reliability analysis methods begin with the prediction of the MPP. The MPP of each limit-state function can be efficiently estimated using the algorithm presented by Wang and Grandhi.[11] This algorithm uses the two-point approximation TANA2 of the actual limit state in the search procedure to reduce the computational time. This method is very efficient when dealing with highly nonlinear implicit problems with a large number of random variables. In the process of searching for the MPP of each limit-state function, a series of data point information, including the function

values and gradients, is obtained. In this research, data points that are obtained in the process of searching for the MPP, as well as the points obtained while estimating the intersection point of the limit states closest to the origin in the normalized domain, are used to construct the local approximations. Once the local approximations are obtained, an MPA is constructed that contains the information of all of the local approximations. The MPA adaptively adjusts itself to behave as a local approximation when a design point is close to one of the data points. Function and gradient values of this MPA correspond directly with their exact counterparts at the points where the local approximations were generated. Monte Carlo simulation is performed on this MPA to obtain the system failure probability.

System Reliability Calculation

1) Estimate the MPP of each limit-state function. The MPP is obtained by using the algorithm presented by Wang and Grandhi.[11]

2) In the process of searching for the MPP of each limit state, the information (function value and gradient) of a number of points on each limit-state function is obtained. With this information, local TANA2 approximations at these design points can be constructed.

3) After obtaining the MPP for each of the limit states, the closest intersection point of all of the limit states in the normalized domain is estimated. During this process, design points are obtained, and these points, along with the intersection point, are used to construct additional local TANA2 approximations.

4) Once the local approximations are constructed, weighting functions that are required to construct the MPA are evaluated. One weighting function is required for each of the local approximations. The weighting function controls the influence of each local approximation at a particular point in the design space. With the same process, an MPA can be constructed for each limit-state function. Figure 2 shows how the MPA adapts to the local approximation at the design points where the local approximation is constructed. When the design point is close to the expansion point of one of the approximations, the weight of that approximation is the maximum, and the MPA takes the shape of that particular local approximation. If the design point is close to the expansion points of two local approximations, the contribution from both of the local approximations is taken care of using the weighting functions.

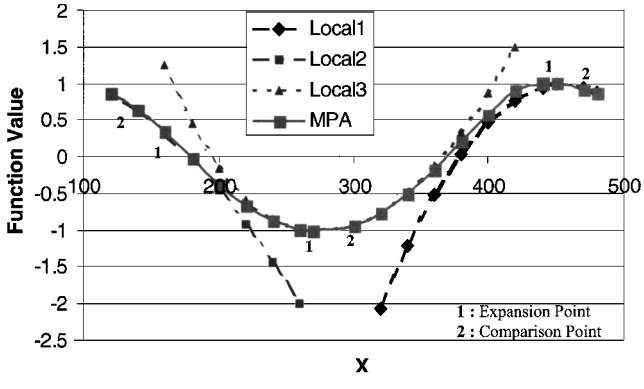

Fig. 2 MPA Representation using TANA2 as local approximations.

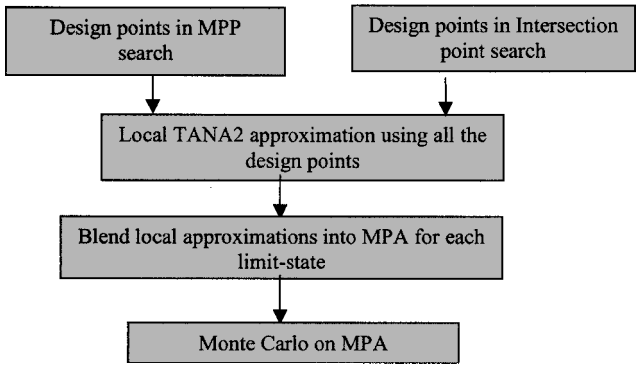

Fig. 3 System reliability calculation.

5) After surrogate representations for each limit-state function are obtained, Monte Carlo simulation is performed on the approximate limit-state functions, which closely represent the actual limit states at the MPP and the data points.

The use of MPA enables the modeling of the n-dimensional joint failure domain for using the Monte Carlo simulation. This approximation reduces a considerable amount of computational effort because the MPA of each limit-state function is explicit, without sacrificing much accuracy. Because information at more points than the one- and two-point approximations is used to construct the MPA of each limit-state function, the MPA is accurate over a larger region. Figure 3 illustrates the methodology just discussed.

Numerical Examples

Three examples are provided to show the applicability of the proposed method. This method can be applied for problems with multiple nonnormal random variables and implicit or explicit limit states. Various examples have been studied to demonstrate the efficiency and accuracy of this method. This method can produce an accurate value of the probability of failure, unlike the other methods that only produce the bounds on its value. The final failure probability of the system estimated by using an MPA is compared with the results obtained directly from a Monte Carlo simulation. Each limit state is approximated using the MPA, and the Monte Carlo simulation is performed on the approximate limit-state functions.

Cantilever Beam

A cantilever beam, shown in Fig. 4, is subjected to a tip load $P = 36.3$ kgf. Three failure modes, the displacement greater than 0.00381 m (failure), stress greater than 703 kg/cm^2 (failure), and the fundamental frequency less than 10 Hz (failure), are considered: Displacement is found by

$$G_1(X) = 4PL^3/Ebh^3 - 0.00381 \leq 0.0$$

stress by

$$G_2(X) = 12PL/bh^2 - 10^4 \leq 0.0$$

and frequency by

$$G_3(X) = 10 - (1.875)^2(EI/\rho AL^4)^{\frac{1}{2}} \leq 0.0$$

where L, b, and h are the length, width, and height of the beam with mean values of 0.762, 0.0217, and 0.0637 m, respectively, and the Young's modulus E is 703,000 kg/cm^2. The length, width, and height of the beam are considered as the random variables, and the standard deviations are $\sigma_L = 0.0762$, $\sigma_b = 0.00203$, and $\sigma_h = 0.00635$ m, respectively. Both L and h are considered as normally distributed and b is considered as a lognormal distribution.

In this example, each of the limit states is approximated using MPAs. To improve the accuracy of the approximations, two additional points are added to each of the MPAs. These two points are the MPP of the other two limit states when one MPA is being constructed. For example, when an MPA is constructed for displacement, the MPP of stress and frequency limit states are added to the approximation. This procedure improves the accuracy of the MPA at each of the three MPPs. The local approximations constructed at the intermediate points (points obtained during the MPP search and the points obtained during the search for the closest common intersection point) are TANA2 for displacement and stress constraints. A first-order approximation is constructed at each of the other two MPPs; then, these local approximations are added to the MPA.

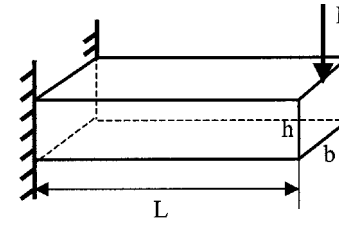

Fig. 4 Cantilever beam.

Table 1 MPA and Monte Carlo results for cantilever beam

Method	System failure probability	Difference, %
Monte Carlo simulation	0.0264	——
MPA ($m = 2.0$)	0.0276	4.54
First-order series bounds	0.0269–0.0429	1.89–62.5

The frequency constraint is not dependent on the width of the beam; therefore, there is no change in the value of the width from the mean, and the gradient is zero. For that reason, a first-order approximation is considered instead of TANA2. These first-order approximations are considered at each of the intermediate design points and the MPPs of the displacement and stress constraints.

Table 1 clearly shows the accuracy of the proposed MPA-based system reliability method compared to the first-order series bounds. The MPA results are quite comparable to the Monte Carlo simulation results, the difference being 4.54%. There are 100,000 Monte Carlo simulations performed on the exact limit-state functions and on the MPAs. The FORM bounds were wide; however, the lower bound was a very good approximation. The reason why the FORM bound failed to include the actual failure probability is that the individual failure probability estimated using the FORM technique was not accurate.

Ten-Bar Truss Structure

The system failure probability of a 10-bar truss, shown in Fig. 5, was calculated in this example. The cross-sectional areas of all of the 10 truss members are lognormal distributions with a mean value of 0.0635 m and a standard deviation of 0.00635 m. The Young's modulus is 703,000 kg/cm^2 and the forces applied are $P_1 = P_2 = 45{,}400$-kg force, as shown in Fig. 5. Two limit states have been considered to estimate the system failure probability. One is a displacement limit, and the other is an eigenvalue limit. The maximum displacement of the tip of the truss structure should be less than 0.0457 m, and the eigenvalue must be greater than 177 (rad/s)^2:

$$G_1(X) = D_{\text{tip}}/0.0457 - 1.0 \geq 0.0$$

$$G_2(X) = 1.0 - \lambda_1/177.0 \geq 0.0$$

The structural analysis is done using ASTROS, an FEA program. Table 2 compares the results obtained by different methods and the Monte Carlo method. Because this is a problem with implicit limit-state functions, TANA2s are constructed at the data points obtained in the process of searching for the MPP and the intersection point search. Seven data points were used for the displacement constraint, four from the MPP search and three from the intersection point search. For the frequency constraint, eight data points were used, five from the MPP search and three from the intersection point search. These local approximations are blended together using the MPA method. There are two MPAs: One corresponds to the displacement limit state, and the other corresponds to the fundamental frequency. In this example, 100,000 simulations using ASTROS were used to estimate the system failure probability using the Monte Carlo method.

With the MPA method, the system failure probability obtained was 0.0058, which is a 7.5% difference from the actual value. The results obtained from the Monte Carlo simulation are the converged results after 100,000 finite element simulations. Each simulation involves evaluation of the displacement and frequency limit-state function values. Therefore, 100,000 simulations involved 200,000 calls to the finite element method package, which in this case was ASTROS. As seen in Table 2, the first-order series bounds are not precise because neither of the limit states are linear functions. Even though additional bounding techniques are available, they require additional computational effort to obtain the bounds. The comparison in this example is carried out between methods that require no additional simulations after the safety index is obtained for all of the limit states. The MPA was accurate, and it was able to integrate the individual failure domains to model the joint failure domain. A good local approximation of the individual limit states ensures good system failure probability estimation.

Table 2 MPA and Monte Carlo results for 10-bar truss

Method	System failure probability	Difference, %
Monte Carlo simulation	0.0054	——
MPA ($m = 2.0$)	0.0058	7.5
First-order series bounds	0.0039–0.0062	−28.3–13.9

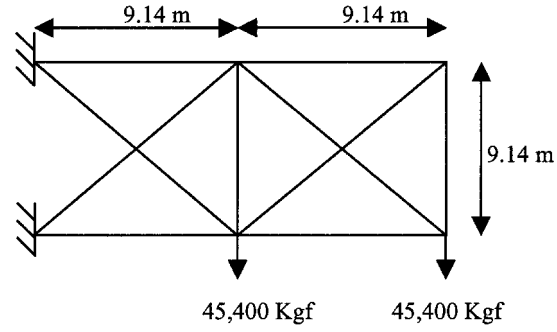

Fig. 5 Truss with 10 bars.

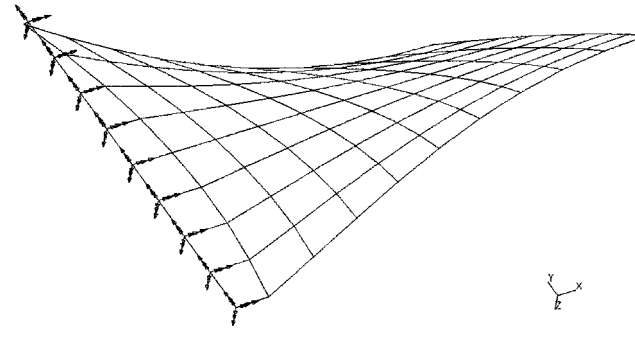

Fig. 6 Turbine blade.

Turbine Blade

The turbine blade geometry is shown in Fig. 6, with a 45-deg twist angle. The blade is modeled with 80 quadrilateral plate bending elements with 99 nodes. All of the degrees of freedom along the hub are fixed. The thicknesses of the plate elements are considered as the random variables, but, with physical linking, only 10 independent random variables are considered. All of the chordwise elements are assumed to have the same thickness. All of the element thicknesses have a mean value of 0.00889 m, with coefficient of variation of 0.10.

Three different limit states are considered; for safety, the blade model must satisfy 1) displacement in the Z direction,

$$D_{\text{tip}}/0.015 - 1.0 \leq 0.0$$

2) first natural frequency,

$$1.0 - \lambda_1/2500 \leq 0.0$$

and 3) stress in element 77 (root),

$$\left[\left(\frac{\sigma_x}{65{,}000}\right)^2 + \left(\frac{\sigma_y}{65{,}000}\right)^2 - \frac{\sigma_x\sigma_y}{65{,}000 \times 65{,}000} - \left(\frac{\tau_{xy}}{9000}\right)^2\right]^{\frac{1}{2}} - 1.0 \leq 0.0$$

The MPP is estimated for all of the limit states; then, when the intermediate points are used in the MPP search algorithm, local approximations are constructed. Once the MPP is obtained, the closest intersection point is obtained by posing it as an optimization problem as follows: Minimize $(\boldsymbol{U}^T\boldsymbol{U})^{1/2}$, subject to

$$g_1(\boldsymbol{U}) = 0, \qquad g_2(\boldsymbol{U}) = 0$$

where $\boldsymbol{U}$ is the vector of random variables in the transformed domain and $g_1(\boldsymbol{U})$ and $g_2(\boldsymbol{U})$ are the two limit-state functions used to determine the intersection point. This optimization problem is solved three times by using two different limit states in each analysis to

Table 3 MPA and Monte Carlo results for turbine blade

Method	System failure probability	Difference, %
Monte Carlo simulation	0.0121	——
MPA ($m = 2.0$)	0.0132	8.59
First-order series bounds	0.0059–0.0125	−51.45–3.55

obtain all of the three intersection points for the three limit states. DOT[12] was used in this research work to perform optimization.

For stress constraint, six actual simulations were required to converge to the MPP; therefore, seven design points were available to construct the local approximations. The displacement constraint converged in seven iterations, resulting in eight design points; therefore, seven TANA2 are used to construct the MPA for the displacement limit state. When the closest intersection point was obtained, two additional intermediate points were obtained. This information was added to the available information, along with the information about the intersection point. For the frequency constraint, four TANA2 were constructed at various design points, and these approximations were blended into a third MPA. The points obtained in the intersection point search were common for all of the three limit states.

Once these limit states were available as closed-form MPAs, Monte Carlo simulation was used to obtain the system failure probability. The results are presented in Table 3. The results obtained from the MPA are compared with the actual Monte Carlo results and FORM results. The first-order bound method could approximate the upper bound of the system failure probability very accurately, to within 4% error. However, the estimated lower bound was highly inaccurate, and this would make the decision making based on the estimated bound a daunting task. The proposed method would efficiently obtain the system failure probability for structural systems that have highly nonlinear limit states with high accuracy. The results show that the system failure probability obtained by the proposed method had an error of 8.59% on the conservative side. A certain percentage of difference is inevitable in between the Monte Carlo estimation and MPA due to the way these methods represent the actual response.

Summary

The computation of system reliability for multiple implicit limit states is a complex and time-consuming task. The use of MPA enables modeling of the n-dimensional joint failure domain for building surrogate models that can be used in a Monte Carlo simulation. Because the MPA of each limit-state function is explicit, this approximation considerably reduces the computational effort without sacrificing much accuracy. Because information at more points is used to construct the MPA of each limit-state function, the MPA is accurate over a larger region compared to the one- and two-point approximations. It is possible to provide a good prediction of the intersection points of different limit-state functions.

By the use of the available methods in the literature, the bounds on the system failure probability can be obtained. These bounds are estimated by using approximation techniques, unlike the n-dimensional integration, which is more accurate. This can lead into an additional uncertainty in the bounds. In MPA, the failure probability of the system is available as a single value, and it takes into account the correlation between the limit states. After solving a class of problems with MPA, it is possible to understand where the result stands in comparison to the Monte Carlo simulation.

MPA has a tremendous potential for problems where the limit states are not unimodal and exhibit high nonlinearity. In those cases, the MPP search starts from a mean point and approaches from one side of the nonlinear surface. The points generated in the search may represent a very small region of the nonlinear domain. If an approximation is built only using the searched failure points, then the system reliability may not include the complete failure region. In cases where TANA2 converges rapidly to the MPP, the entire domain may not have been investigated. In those situations, a design of experiments approach for choosing the additional points for building several local approximations is appropriate. The idea is to capture the failure region accurately using multiple approximations.

Appendix: MPA Based on Local Approximations

The MPA[10] can be regarded as the connection of many local approximations. With function and sensitivity information already available at a series of points, one local approximation is built at each point. All local approximations are then integrated into an MPA by the use of a weighting function. The weighting functions are selected such that the approximation reproduces function and gradient information at the known data points.

The local approximations discussed in this section are TANA2s.[9] The function $F(\boldsymbol{X})$ and gradient $\partial F(\boldsymbol{X})/\partial x$ information is available at

$$\boldsymbol{X}_k = (x_{1,k}, x_{2,k}, \ldots, x_{n,k})^T, \qquad k = 1, 2, \ldots, K$$

The MPA can be written in terms of the local approximations as

$$\tilde{F}(\boldsymbol{X}) = \sum_{k=1}^{K} W_k(\boldsymbol{X})\,\tilde{F}_k(\boldsymbol{X}) \tag{A1}$$

where W_k is a weighting function

$$W_k(\boldsymbol{X}) = \frac{\phi_k(\boldsymbol{X})}{\sum_{j=1}^{K} \phi_j(\boldsymbol{X})} \tag{A2}$$

and $\tilde{F}_k(\boldsymbol{X})$ is the TANA2. $W_k(\boldsymbol{X})$ adjusts the contribution of $\tilde{F}_k(\boldsymbol{X})$ to $\tilde{F}(\boldsymbol{X})$ in Eq. (A1). Here, $\phi_k(\boldsymbol{X})$ is called a blending function and has its maximum of 1 at $\boldsymbol{X}_k$ and vanishes when $\boldsymbol{X}_k$ is very far from $\boldsymbol{X}$.

The important details of the TANA2[9] are presented hereafter. Further details can be found in Ref. 6. The physical variables are transformed to the intervening variables using the relation

$$y_i = x_i^{p_i}, \qquad i = 1, 2, \ldots, n$$

where the exponents p_i represent the nonlinear indices and are different for each variable, y_i is the intervening variable, and x_i is the physical variable. Information at two points, namely, the comparison point X_1 and the expansion point X_2, is used in building the approximation. The approximation is obtained by expanding the function at the expansion point X_2 as

$$\tilde{g}(X) = g(X_2) + \sum_{i=1}^{n} \frac{\partial g(X_2)}{\partial x_i} \frac{x_{i,2}^{1-p_i}}{p_i}\left(x_i^{p_i} - x_{i,2}^{p_i}\right) + \frac{1}{2}\varepsilon \sum_{i=1}^{n} \left(x_i^{p_i} - x_{i,2}^{p_i}\right)^2 \tag{A3}$$

This equation is a second-order Taylor series expansion in terms of the intervening variables, in which the Hessian matrix has only diagonal elements of the same value ε. Therefore, this approximation does not need the calculation of second-order derivatives. The error from the approximate Hessian matrix is partially corrected by adjusting the nonlinearity index p_i. In contrast to the true quadratic approximation, this approximation is closer to the actual function for highly nonlinear problems due to its adaptability.

Equation (A3) has $n + 1$ unknown constants, so that $n + 1$ equations are required. When Eq. (A3), is differentiated n equations are obtained by matching the derivatives available at the previous point X_1:

$$\frac{\partial g(X_1)}{\partial x_i} = \left(\frac{x_{i,1}}{x_{i,2}}\right)^{p_i - 1} \frac{\partial g(X_2)}{\partial x_i} + \varepsilon\left(x_i^{p_i} - x_{i,2}^{p_i}\right) x_{i,1}^{p_i - 1} p_i \qquad i = 1, 2, \ldots, n \tag{A4}$$

Another equation is obtained by matching the exact and approximate function values with the previous point X_1:

$$g(X_1) = g(X_2) + \sum_{i=1}^{n} \frac{\partial g(X_2)}{\partial x_i} \frac{x_{i,2}^{1-p_i}}{p_i}\left(x_{i,1}^{p_i} - x_{i,2}^{p_i}\right) + \frac{1}{2}\varepsilon \sum_{i=1}^{n} \left(x_{i,1}^{p_i} - x_{i,2}^{p_i}\right)^2 \tag{A5}$$

In this method, the exact function and derivative values are equal to the approximate function and derivative values, respectively, at the previous and current points.

Several blending functions in Eq. (A2) can used to make the MPA reproduce the exact function and gradient values at the data points where the local approximation was built. There are at least three blending functions that could meet this requirement. They are

$$\phi_k(\boldsymbol{X}) = 1/[\exp(h_k) - 1] \tag{A6}$$

$$\phi_k(\boldsymbol{X}) = 1/\log(h_k + 1) \tag{A7}$$

$$\phi_k(\boldsymbol{X}) = 1/h_k \tag{A8}$$

where

$$h_k = \left[\sum_{i=1}^{n} (x_i - x_{i,k})^2 \right]^m \tag{A9}$$

and where m is a positive integer. Additionally, it is recommended, from computational considerations, that the design space be normalized as $x_i \in [0, 1]$ to measure the weighting function. Equation (A8) was used in this work.

With each of Eqs. (A6–A9), the weighting function of Eq. (A2) has the properties

$$W_k(\boldsymbol{X}_j) = \delta_{kj}, \qquad 0 \le W_k(\boldsymbol{X}_j) \le 1 \tag{A10}$$

$$\lim_{x_i \to \pm\infty} W_k(\boldsymbol{X}) = \frac{1}{K} \tag{A11}$$

$$\sum_{k=1}^{K} W_k(\boldsymbol{X}) = 1 \tag{A12}$$

The weighting function varies between 0 and 1, and the summation of all weighting functions is 1. The following properties can be shown for each blending function given in Eqs. (A6–A8):

$$\frac{\partial W_k(\boldsymbol{X}_j)}{\partial x_i} = 0 \tag{A13}$$

When Eq. (1) is differentiated,

$$\frac{\partial \tilde{F}(\boldsymbol{X})}{\partial x_i} = \sum_{k=1}^{K} \left[\frac{\partial W_k(\boldsymbol{X})}{\partial x_i} \tilde{F}_k(\boldsymbol{X}) + W_k(\boldsymbol{X}) \frac{\partial \tilde{F}_k(\boldsymbol{X})}{\partial x_i} \right] \qquad i = 1, 2, \ldots, n \tag{A14}$$

From Eqs. (A1), (A10), (A13), and (A14), the following are obtained:

$$\tilde{F}(\boldsymbol{X}_j) = \tilde{F}_j(\boldsymbol{X}_j) = F(\boldsymbol{X}_j), \qquad j = 1, 2, \ldots, K \tag{A15}$$

$$\frac{\partial \tilde{F}(\boldsymbol{X}_j)}{\partial x_i} = \frac{\partial \tilde{F}_j(\boldsymbol{X}_j)}{\partial x_i} = \frac{\partial F(\boldsymbol{X}_j)}{\partial x_i} \qquad i = 1, 2, \ldots, n, \qquad j = 1, 2, \ldots, K \tag{A16}$$

Equations (A15) and (A16) show that the MPA has the same zero-order and first-order information as the original function at the data points:

$$\lim_{x_i \to \pm\infty} \tilde{F}(\boldsymbol{X}) = \frac{1}{K} \sum_{k=1}^{K} \tilde{F}_k(\boldsymbol{X}) \tag{A17}$$

The MPA is an average value of all TANA2 estimations when a design point is far from every data point.

Reliability analysis involves iterations, which require implicit function evaluations and gradient evaluations that are expensive and come from finite element simulation. Therefore, the use of approximations helps reduce the cost involved in each analysis, without sacrificing the accuracy of the results. Multipoint function approximations are more suitable for system reliability analysis where the failure domain is the result of multiple failure criteria because each of the failure surfaces has to be approximated separately and then combined to represent the failure domain. The two-point approximation (TANA2) discussed earlier is used as the local approximation.

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